The Fourier Transform: Examples, Properties, Common Pairs

CS 450: Introduction to Digital Signal and Image Processing

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Magnitude and Phase

Remember: complex numbers can be thought of as (real,imaginary) or (magnitude,phase).

$$\begin{array}{lcl} \text{Magnitude:} & |F| & = & \left[\Re(F)^2 + \Im(F)^2\right]^{1/2} \\ \text{Phase:} & \phi(F) & = & \tan^{-1} \frac{\Im(F)}{\Re(F)} \end{array}$$

Real part	How much of a cosine of that frequency you need
Imaginary part	How much of a sine of that frequency you need
Magnitude	Amplitude of combined cosine and sine
Phase	Relative proportions of sine and cosine

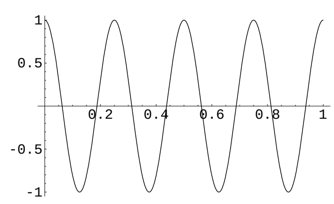
Example: Fourier Transform of a Cosine

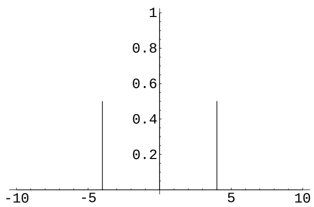
$$f(t) = \cos(2\pi st)$$

$$\begin{aligned} F(u) &= \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi ut}\, dt \\ &= \int_{-\infty}^{\infty} \cos(2\pi st)\, e^{-i2\pi ut}\, dt \\ &= \int_{-\infty}^{\infty} \cos(2\pi st)\, [\cos(-2\pi ut) + i \sin(-2\pi ut)]\, dt \\ &= \int_{-\infty}^{\infty} \cos(2\pi st)\, \cos(-2\pi ut)\, dt + i \int_{-\infty}^{\infty} \cos(2\pi st)\, \sin(-2\pi ut)\, dt \\ &= \underbrace{\int_{-\infty}^{\infty} \cos(2\pi st)\, \cos(2\pi ut)\, dt}_{0 \text{ except when } u = \pm s} - i \underbrace{\int_{-\infty}^{\infty} \cos(2\pi st)\, \sin(2\pi ut)\, dt}_{0 \text{ for all } u} \\ &= \frac{1}{2}\delta(u-s) + \frac{1}{2}\delta(u+s) \end{aligned}$$

Example: Fourier Transform of a Cosine

Spatial Domain	Frequency Domain
$\cos(2\pi st)$	$\frac{1}{2}\delta(u-s) + \frac{1}{2}\delta(u+s)$





Odd and Even Functions

Even	Odd
$f(-t) = f(t)$	$f(-t) = -f(t)$
Symmetric	Anti-symmetric
Cosines	Sines
Transform is real*	Transform is imaginary*

* for real-valued signals

Sinusoids

Spatial Domain $f(t)$	Frequency Domain $F(u)$
$\cos(2\pi st)$	$\frac{1}{2}\left[\delta(u+s) + \delta(u-s)\right]$
$\sin(2\pi st)$	$\frac{1}{2} i \left[\delta(u+s) - \delta(u-s)\right]$

Constant Functions

Spatial Domain $f(t)$	Frequency Domain $F(u)$
1	$\delta(u)$
a	$a\ \delta(u)$

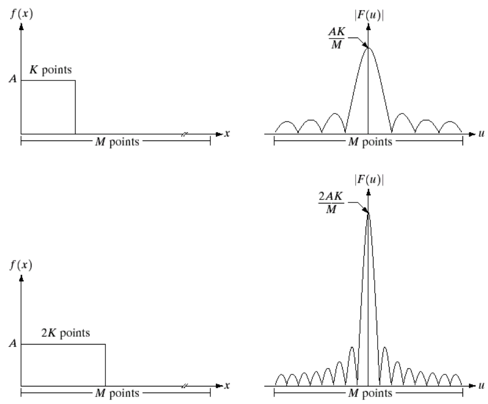
Delta Functions

Spatial Domain $f(t)$	Frequency Domain $F(u)$
$\delta(t)$	1

Square Pulse

Spatial Domain $f(t)$	Frequency Domain $F(u)$
$\begin{cases} 1 & \text{if } -a/2 \le t \le a/2 \\ 0 & \text{otherwise} \end{cases}$	$\text{sinc}(a\pi u) = \frac{\sin(a\pi u)}{a\pi u}$

Square Pulse



a b
c d

FIGURE 4.2 (a) A discrete function of M points, and (b) its Fourier spectrum. (c) A discrete function with twice the number of nonzero points, and (d) its Fourier spectrum.

Triangle

Spatial Domain $f(t)$	Frequency Domain $F(u)$
$\begin{cases} 1 - \lvert t \rvert & \text{if } -a \le t \le a \\ 0 & \text{otherwise} \end{cases}$	$\text{sinc}^2(a\pi u)$

Comb

Spatial Domain $f(t)$	Frequency Domain $F(u)$
$\delta(t \bmod k)$	$\delta(u \bmod 1/k)$

Gaussian

Spatial Domain $f(t)$	Frequency Domain $F(u)$
$e^{-\pi t^2}$	$e^{-\pi u^2}$

Differentiation

Spatial Domain $f(t)$	Frequency Domain $F(u)$
$\frac{d}{dt}$	$2\pi iu$

Some Common Fourier Transform Pairs

Spatial Domain $f(t)$		Frequency Domain $F(u)$	
Cosine	$\cos(2\pi st)$	Deltas	$\frac{1}{2}[\delta(u+s)+\delta(u-s)]$
Sine	$\sin(2\pi st)$	Deltas	$\frac{1}{2}i[\delta(u+s)-\delta(u-s)]$
Unit	1	Delta	$\delta(u)$
Constant	a	Delta	$a\delta(u)$
Delta	$\delta(t)$	Unit	1
Comb	$\delta(t \bmod k)$	Comb	$\delta(u \bmod 1/k)$

More Common Fourier Transform Pairs

Spatial Domain $f(t)$		Frequency Domain $F(u)$	
Square	1 if $-a/2 \leq t \leq a/2$ 0 otherwise	Sinc	$\mathrm{sinc}(a\pi u)$
Triangle	$1-\lvert t\rvert$ if $-a \leq t \leq a$ 0 otherwise	Sinc^2	$\mathrm{sinc}^2(a\pi u)$
Gaussian	$e^{-\pi t^2}$	Gaussian	$e^{-\pi u^2}$
Differentiation	$\frac{d}{dt}$	Ramp	$2\pi iu$

Properties: Notation

Let $\mathcal{F}$ denote the Fourier Transform:

$$F = \mathcal{F}(f)$$

Let $\mathcal{F}^{-1}$ denote the Inverse Fourier Transform:

$$f = \mathcal{F}^{-1}(F)$$

Properties: Linearity

Adding two functions together adds their Fourier Transforms together:

$$\mathcal{F}(f+g) = \mathcal{F}(f) + \mathcal{F}(g)$$

Multiplying a function by a scalar constant multiplies its Fourier Transform by the same constant:

$$\mathcal{F}(af) = a\ \mathcal{F}(f)$$

Properties: Translation

Translating a function leaves the magnitude unchanged and adds a constant to the phase.
If

$$\begin{aligned} f_2 &= f_1(t-a) \\ F_1 &= \mathcal{F}(f_1) \\ F_2 &= \mathcal{F}(f_2) \end{aligned}$$

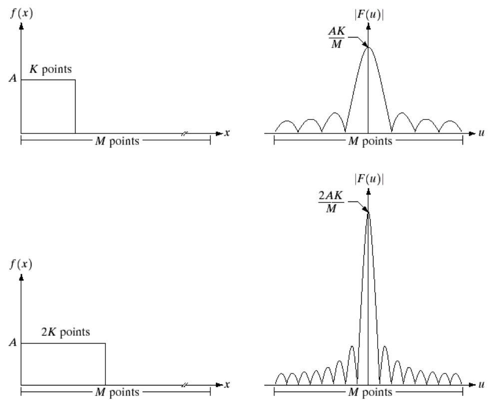
then

$$\begin{aligned} |F_2| &= |F_1| \\ \phi(F_2) &= \phi(F_1) - 2\pi ua \end{aligned}$$

Intuition: magnitude tells you "how much", phase tells you "where".

The Fourier Transform: Examples, Properties, Common Pairs

Change of Scale: Square Pulse Revisited



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Rayleigh's Theorem

Total "energy" (sum of squares) is the same in either domain:

$$\int_{-\infty}^{\infty} |f(t)|^2 \; dt = \int_{-\infty}^{\infty} |F(u)|^2 \; du$$